



Introduction to Data Science

What Do These Matrix Operations Actually Mean?

15 Sep 2026

Eigenvectors and Eigenvalues

Special Vectors Under a Transformation

$$v \rightarrow A v = c v$$

EIGENVECTOR

A nonzero vector with this special transformation property.

EIGENVALUE

The scalar that tells us how the vector is scaled.

We now consider square matrices, so the input and output have the same dimension.

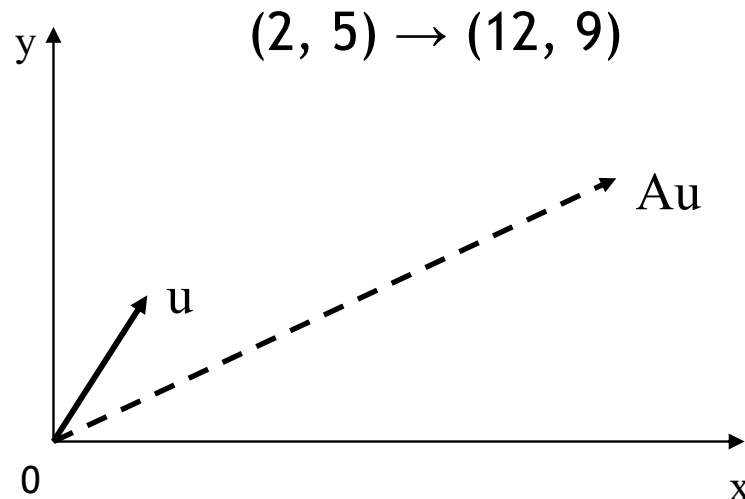
Some vectors are transformed into scalar multiples of themselves.

Eigenvectors and Eigenvalues

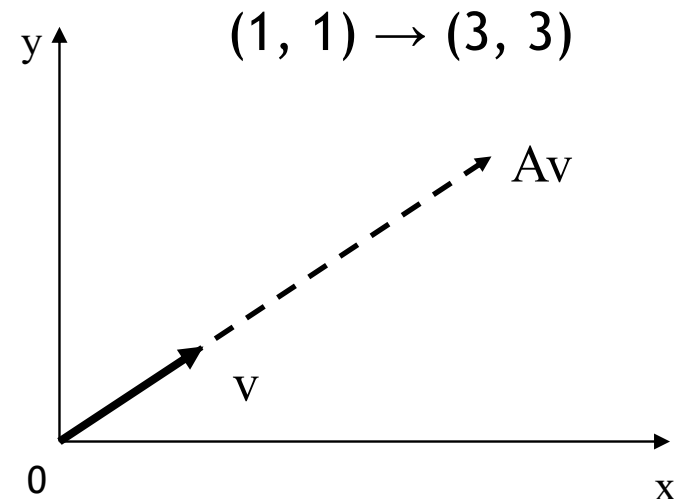
What Happens When a Matrix Transforms a Vector?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

ORDINARY VECTOR



SPECIAL VECTOR



A matrix can change direction, but some vectors only change by a scalar factor.

Eigenvectors and Eigenvalues

Eigenvector

$$A v = c v, \quad v \neq 0$$

Eigenvalue

What happens to the eigenvector?

$$c > 0$$

The output points in the same direction.

$$c < 0$$

The output points in the opposite direction.

$$c = 0$$

The vector is mapped to zero.

The vector stays on the same line through the origin, or collapses to zero.

An eigenvector is nonzero and is mapped to a scalar multiple of itself.

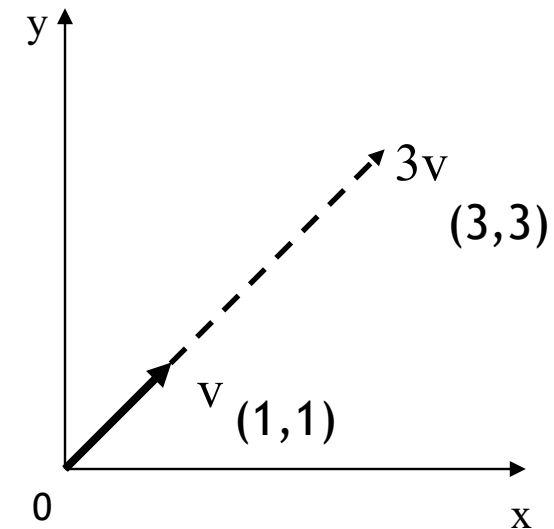
Eigenvectors and Eigenvalues

Example

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The output is exactly three times the input.

There is no change in direction because c is positive.



For this matrix, $(1, 1)$ is an eigenvector with eigenvalue 3.

Eigenvectors and Eigenvalues

Identity Matrix Revisited

$$I v = v = 1 v$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I(1, 1) = (1, 1)$$

$$I(2, 5) = (2, 5)$$

The identity matrix changes neither the values nor the direction of a vector.

Every nonzero vector is an eigenvector of I with eigenvalue 1.

Eigenvectors and Eigenvalues

How Do We Find Eigenvalues?

$$Av = cv, \quad v \neq 0$$

Symbol	Meaning	Known or unknown?
A	The square transformation matrix	Known
v	A nonzero input vector	Unknown
c	The scale factor for that vector	Unknown

Earlier example: $A(1, 1) = 3(1, 1)$. How can we find 3 without guessing?

We want a nonzero vector whose output equals a scalar multiple of itself.

Eigenvectors and Eigenvalues

Why Does the Identity Matrix Appear?

$$Av = cv$$

Start with the eigenvector definition.

$$Av = cIv$$

$Iv = v$, so inserting I changes nothing.

$$Av - cIv = 0$$

Subtract the same vector from both sides.

$$(A - cI)v = 0$$

Factor out the common vector v .

The identity matrix lets us combine both terms into one matrix expression.

Eigenvectors and Eigenvalues

Why Must the Vector Be Nonzero?

$$B = A - cI$$

The zero vector always works:

$$B0 = 0$$

This gives no special direction and puts no restriction on c .

A SEPARATE MATRIX EXAMPLE

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M(0, 1) = (0, 0)$$

The input $(0, 1)$ is nonzero.
 M erases the vertical component.

A nontrivial null space contains at least one nonzero vector.

Eigenvectors and Eigenvalues

Why Must $\det(A - cI)$ Equal Zero?

$$Bv = 0, \quad B = A - cI$$

IF B HAD AN INVERSE

$$B^{-1}Bv = B^{-1}0$$

$$Iv = 0, \quad \text{so } v = 0$$

This conflicts with $v \neq 0$.

WHAT WE NEED

B must have no inverse.

For a square matrix, this means:

$$\det(B) = 0$$

$$\det(A - cI) = 0$$

For a square matrix, $\det(B) = 0$ exactly when $Bv = 0$ has a nonzero solution.

Eigenvectors and Eigenvalues

The Determinant of a 2×2 Matrix

$$B = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \quad \det(B) = ps - qr$$

Main diagonal product minus the other diagonal product.

A MATRIX THAT FLATTENS THE PLANE

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad \det(B) = 2 \times 0 - 0 \times 0 = 0$$

$B(x, y) = (2x, 0)$. Every output lies on one line.

For this 2×2 notation, the determinant is $ps - qr$.

Eigenvectors and Eigenvalues

Worked Example: Building $A - cI$

THE ORIGINAL MATRIX

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

THE SCALED IDENTITY

$$cI = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

$$A - cI = \begin{bmatrix} 1 - c & 2 \\ 2 & 1 - c \end{bmatrix}$$

The 0 entries of cI leave the off-diagonal values unchanged.

Subtract c from each diagonal entry and keep the other entries unchanged.

Eigenvectors and Eigenvalues

Worked Example: The Characteristic Equation

$$\det(A - cI) = (1 - c)(1 - c) - 2 \times 2$$

$$= (1 - c)^2 - 4$$

$$= 1 - 2c + c^2 - 4$$

$$= c^2 - 2c - 3$$

$$A - cI =$$

$$\begin{bmatrix} 1 - c & 2 \\ 2 & 1 - c \end{bmatrix}$$

We therefore solve: $c^2 - 2c - 3 = 0$.

Setting the characteristic polynomial equal to zero gives an equation for c .

Eigenvectors and Eigenvalues

Worked Example: Solving for c

$$c^2 - 2c - 3 = (c - 3)(c + 1)$$

$$(c - 3)(c + 1) = 0$$

A product equals zero when at least one factor equals zero.

$$c - 3 = 0$$

$$c + 1 = 0$$

$$c = 3$$

$$c = -1$$

The eigenvalues of A are 3 and -1 .

Eigenvectors and Eigenvalues

Finding the Corresponding Eigenvectors

$$(A - cI)v = 0$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

v_1 is the first coordinate.

v_2 is the second coordinate.

Eigenvalue	Equation to solve
$c = 3$	$(A - 3I)v = 0$
$c = -1$	$(A + I)v = 0$

Each eigenvalue gives a new linear system for the components of v .

Eigenvectors and Eigenvalues

Eigenvector for $c = 3$

$$A - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

THE TWO ROW EQUATIONS

$$-2v_1 + 2v_2 = 0$$

$$2v_1 - 2v_2 = 0$$

Both rows give the same condition.

$$v_2 = v_1$$

Choose $v = (1, 1)$

The equation $v_2 = v_1$ describes the line through $(1, 1)$.

Eigenvectors and Eigenvalues

Checking the Eigenvector for $c = 3$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

FIRST COORDINATE

$$1 \times 1 + 2 \times 1 = 3$$

SECOND COORDINATE

$$2 \times 1 + 1 \times 1 = 3$$

$A(1, 1) = 3(1, 1)$, so the direction stays the same and the length triples.

Eigenvectors and Eigenvalues

Eigenvector for $c = -1$

$$A - (-1)I = A + I$$

$$A + I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

BOTH ROWS GIVE

$$2v_1 + 2v_2 = 0$$

$$v_2 = -v_1$$

Choose $v = (1, -1)$

The equation $v_2 = -v_1$ describes the line through $(1, -1)$.

Eigenvectors and Eigenvalues

Checking the Eigenvector for $c = -1$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$1 \times 1 + 2 \times (-1) = -1$$

$$2 \times 1 + 1 \times (-1) = 1$$

The direction reverses. The length stays the same because $|-1| = 1$.

A negative eigenvalue reverses the arrow along the same line.

Eigenvectors and Eigenvalues

One Direction, Many Eigenvectors

$$v = t(1, 1), \quad t \neq 0$$

Input v	Output Av	Relationship
$(1, 1)$	$(3, 3)$	$Av = 3v$
$(2, 2)$	$(6, 6)$	$Av = 3v$
$(-1, -1)$	$(-3, -3)$	$Av = 3v$

For $c = -1$, the corresponding vectors are $t(1, -1)$, with $t \neq 0$.

A nonzero multiple of an eigenvector has the same eigenvalue.

Eigenvectors and Eigenvalues

Eigenvalues and Eigenvectors: Recap

Eigenvalue	Example eigenvector	Effect of A
3	(1, 1)	Same direction, three times as long
-1	(1, -1)	Opposite direction, same length

- First, solve $\det(A - cI) = 0$ to find the possible values of c .
- Then solve $(A - cI)v = 0$ for each c , keeping $v \neq 0$.
- Finally, check the result by verifying $Av = cv$.

The eigenvalue gives the scale factor. The eigenvector gives a special direction.

Eigenvectors and Eigenvalues

PCA: Why Use Fewer Coordinates?

PRINCIPAL COMPONENT ANALYSIS (PCA)

Each student has two test scores:

$$\mathbf{x} = (\text{Test 1 score}, \text{Test 2 score})$$

If the scores tend to rise together, they share a common pattern.

One new coordinate can summarize much of this pattern.

PCA builds that coordinate as a weighted combination of the features.

PCA summarizes data using directions that retain a large amount of variation.

Eigenvectors and Eigenvalues

PCA: Centering and Scaling the Features

CENTERING: SUBTRACT THE MEAN

Original scores	Mean	Centered scores
60, 70, 80	70	-10, 0, 10

STANDARDIZING: DIVIDE BY THE STANDARD DEVIATION

$$z = (\text{value} - \text{mean}) / \text{standard deviation}$$

Each standardized feature has mean 0 and variance 1.

Eigenvectors and Eigenvalues

PCA: A Correlation Matrix

$$C = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

Diagonal entries: each feature has correlation 1 with itself.

Off-diagonal entries: the correlation between the two features is 0.8.

The two features tend to increase together.

C summarizes how the standardized features vary together.

Eigenvectors and Eigenvalues

PCA: Eigenvectors and Variance

Principal direction	Unit eigenvector	Eigenvalue
Features increase together	$u_1 = (1, 1) / \sqrt{2}$	1.8
One feature rises as the other falls	$u_2 = (1, -1) / \sqrt{2}$	0.2

Variance measures spread along a direction.

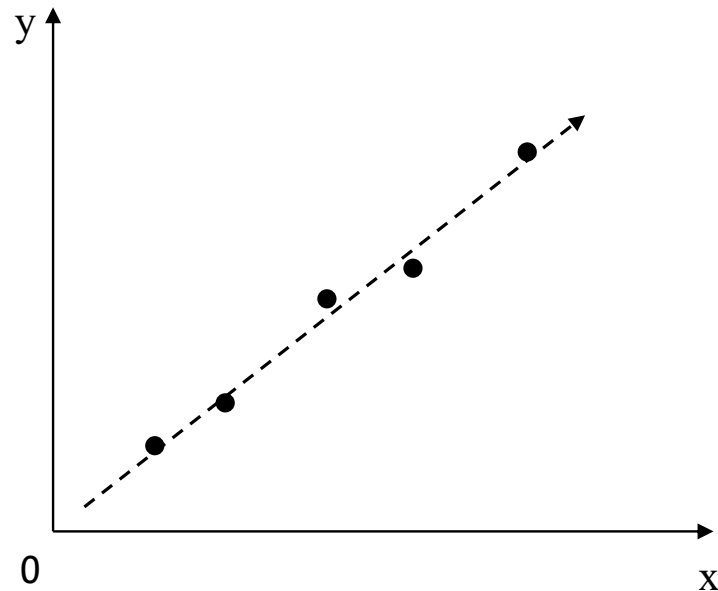
A unit eigenvector has length 1.

For PCA, an eigenvalue measures variance along its unit eigenvector.

Eigenvectors and Eigenvalues

PCA: Projection onto One Direction

TWO FEATURES (SCHEMATIC)



Most variation follows one line.

ONE PRINCIPAL COORDINATE



Project each point onto the line.
Keep its position along that line.

The perpendicular offset is lost.

Projection replaces two coordinates with one position along a selected line.

Eigenvectors and Eigenvalues

PCA: Computing One New Coordinate

$$\mathbf{z} = (2, 0)$$

$$\mathbf{u}_1 = (1, 1) / \sqrt{2}$$

$\mathbf{z}$ is a standardized data point. $\mathbf{u}_1$ is the selected unit direction.

$$\begin{aligned} t = \mathbf{u}_1^T \mathbf{z} &= (1/\sqrt{2}) \times 2 + (1/\sqrt{2}) \times 0 \\ &= \sqrt{2} \approx 1.41 \end{aligned}$$

The new coordinate is a dot product with the selected unit eigenvector.

Eigenvectors and Eigenvalues

PCA: What Do We Keep and Lose?

Original point: $z = (2, 0)$

Reconstruction: $\hat{z} = t u_1 = (1, 1)$

Omitted part: $z - \hat{z} = (1, -1)$

We keep the position along the selected line.

We lose the offset perpendicular to that line.

Fewer coordinates simplify the data, but discarded directions can still matter.

Summary

Quick Check: Why Is a Dot Product One Number?

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \times 4 + 2 \times 5 + 3 \times 6$$
$$= 4 + 10 + 18 = 32$$

Three products enter the calculation.
Adding them produces one number.

A row vector times a matching column vector has shape 1×1 .

Summary

Quick Check: Column Space and Null Space

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B(x, y) = (x, 0)$$

Space	Which vectors?	For this B
Column space	Every reachable output	$(x, 0)$: the horizontal axis
Null space	Every input that becomes zero	$(0, y)$: the vertical axis

Column space: reachable outputs. Null space: inputs that become zero.

Summary

Quick Check: Is $Av = -v$ an Eigenvector Equation?

$$Av = -v = (-1)v, \quad v \neq 0$$

The output points in the opposite direction on the same line.
Its length stays the same.

ONE MORE DISTINCTION

An eigenvalue can be 0: a nonzero vector may satisfy $Av = 0$.
The zero vector itself is never an eigenvector.

Yes, when $v \neq 0$. The eigenvalue is -1 .

Summary

What Did We Learn?

Topic	Plain-language meaning
Linear combination	Scale vectors and add them.
Column space	All outputs that a matrix can generate.
Null space	All inputs that a matrix sends to zero.
Eigenvector and eigenvalue	A special direction and its scale factor.
PCA	New coordinates that retain selected variation.

Neural networks also use matrix operations to transform data representations.

Linear algebra describes how transformations change the representation of data.