



Introduction to Data Science

What Do These Matrix Operations Actually Mean?

10 Sep 2026

Data Structures and Operations

Overview

1. Compute

Use vectors and matrices to produce numerical outputs.

2. Interpret

Ask which outputs are reachable and which inputs disappear.

3. Connect

Use special directions to understand structure in data.

From numerical operations to the meaning of a transformation.

Data Structures and Operations

Vector Operations

One vector can describe one observation using several numerical values.

ROW FORM

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

COLUMN FORM

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The entries have the same order in both forms.

Next, we use these forms to compare vectors and create matrices.

A vector is an ordered collection of numerical values.

Vector Operations

Row Vector vs. Column Vector

ROW VECTOR

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$1 \times 3$$

One row and three columns.

COLUMN VECTOR

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$3 \times 1$$

Three rows and one column.

In general, the shapes are $1 \times n$ and $n \times 1$. Check the shape before multiplying.

The values stay the same, but the shape changes how we multiply.

Vector Operations

Dot Product

$$\begin{bmatrix} 4 & 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}$$

SAME NUMBER OF ENTRIES

Match the first values, then the second, then the third.

Adding the three products gives one scalar, not another vector.

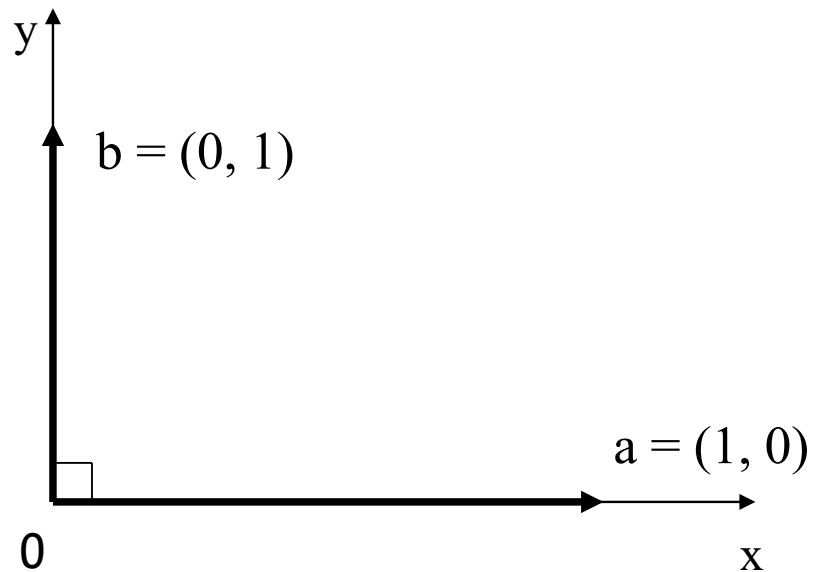
$$4 \times 2 + 3 \times 7 + 6 \times 1 = 8 + 21 + 6 = 35$$

The shape rule is $(1 \times 3)(3 \times 1) \rightarrow 1 \times 1$.

Multiply corresponding values, then add the products.

Vector Operations

Orthogonality



$$a \cdot b = 1 \times 0 + 0 \times 1 = 0$$

The vectors point along the x-axis and the y-axis.

Their directions are perpendicular, so we call them orthogonal.

The zero result describes a relationship between the two directions.

For nonzero real vectors, a zero dot product means perpendicular directions.

Vector Operations

Why Does Dot Product Matter in AI?

TEXT AND TEXT

“A cat sits on a mat.”

“A kitten on a mat.”

$(0.2, 0.8, -0.1, 0.5)$

$\cdot (0.3, 0.7, -0.2, 0.4) = 0.84$

IMAGE AND TEXT

Image embedding

“A dog in the park.”

$(0.1, 0.6, 0.3, -0.4)$

$\cdot (0.0, 0.5, 0.8, -0.1) = 0.58$

These illustrative vectors show how a comparison score is calculated.

With unit-length vectors, the dot product equals cosine similarity.

Dot products provide comparison scores in a shared representation space.

Vector Operations

Outer Product

$$\begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 6 & 12 \\ 28 & 21 & 42 \\ 4 & 3 & 6 \end{bmatrix}$$

Each row is a scaled copy of the row vector (4, 3, 6).

The first row is multiplied by 2, the second by 7, and the third by 1.

$$(3 \times 1)(1 \times 3) \rightarrow 3 \times 3$$

A column vector times a row vector produces a matrix of pairwise products.

Data Structures and Operations

Matrix-Vector Multiplication

$A v = \text{output vector}$

1. Check the shape

The number of columns in A must match the number of values in v .

2. Compute the output

Use row-wise dot products, or multiply each column by its coefficient and add.

The vector tells us how to combine the columns of the matrix.

Matrix-Vector Multiplication

Matrix-Vector Multiplication

$$\begin{bmatrix} 4 & 6 \\ 2 & 7 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 1 \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} + 5 \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 34 \\ 37 \\ 45 \end{bmatrix}$$

The first coefficient is 1, so the first column stays unchanged.

Multiply every entry in the second column by its coefficient, 5.

Add the two scaled columns to obtain (34, 37, 45).

Scale each column by its coefficient, then add the scaled columns.

Matrix-Vector Multiplication

The Same Computation, Two Views

ROW VIEW

$$4 \times 1 + 6 \times 5 = 34$$

$$2 \times 1 + 7 \times 5 = 37$$

$$5 \times 1 + 8 \times 5 = 45$$

Each row gives one output value.

COLUMN VIEW

$$v = (1, 5)$$

Take the first column once.
Take the second column five times.
Add the two resulting vectors.

The columns together give the whole output vector.

Row-wise computation and column-wise interpretation give the same result.

Matrix-Vector Multiplication

Why Does This Matter for Neural Networks?

$$y = W x + b$$

ONE OUTPUT UNIT

x is the input vector.

$$x = (2, 7, 1)$$

W contains learned weights.

$$w = (4, 3, 6), \quad b = 1$$

b is the bias.

y is the affine output.

$$y = 35 + 1 = 36$$

First compute the weighted sum, then add the bias.

A full layer can apply an activation after this affine step.

A neural network repeatedly combines inputs using learned weights.

Summary

What Did We Learn?

Concept	What we can now do
Matrix	Organize numerical values in rows and columns.
Matrix operations	Check shapes before combining matrices.
Vector operations	Distinguish a scalar dot product from an outer-product matrix.
Matrix-vector multiplication	Calculate an output and interpret it as a column combination.
Neural networks	Recognize the weighted computation inside a layer.

We can compute the output. Next, we ask what the computation means.

The Fundamental Spaces

From Computation to Interpretation

$$v \rightarrow A v$$

THE COLUMN SPACE

Which output vectors can A generate?

$C(A)$

Vary the input and collect all possible outputs.

THE NULL SPACE

Which input vectors does A send to zero?

$N(A)$

Look for inputs whose output is the zero vector.

Column space describes outputs. Null space describes inputs mapped to zero.

The Fundamental Spaces

Column Space

$C(A)$ contains all linear combinations of the columns of A .

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

LET s AND t VARY

Every pair (s, t) is reachable. These columns span the whole 2D plane.

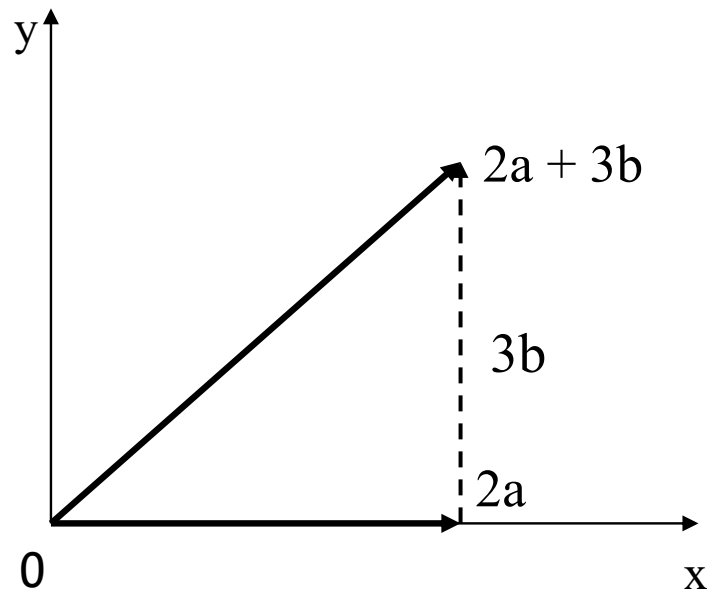
One input gives one output. All possible inputs give the whole column space.

This matrix can generate any vector in the 2D plane.

The column space contains every output that A can generate.

The Column Space

What is a Linear Combination?



$$a = (1, 0), \quad b = (0, 1)$$

$$2a = (2, 0)$$

$$3b = (0, 3)$$

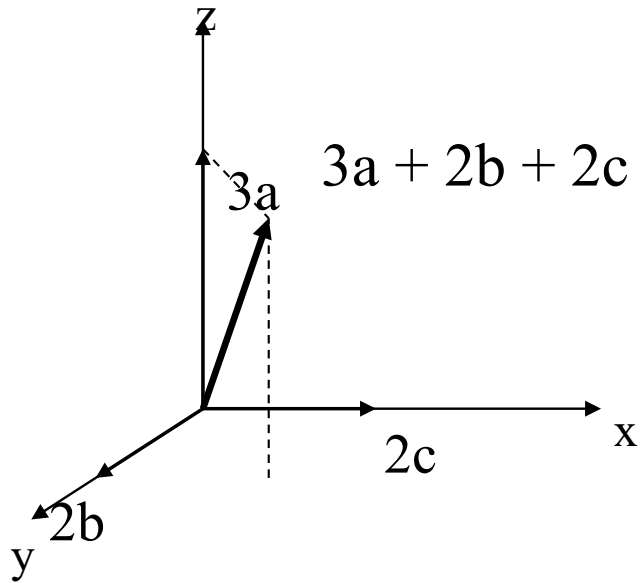
$$2a + 3b = (2, 3)$$

Move twice along a and three times along b . Adding these movements gives $(2, 3)$.

A linear combination is a sum of vectors multiplied by coefficients.

The Column Space

Linear Combinations in Three Dimensions



$$c_1a + c_2b + c_3c$$

First form $3a$.

Then form $2b + 2c$.

Add them to obtain $3a + 2b + 2c$.

The coefficients may be positive, negative, or zero.

We still use the same original vectors, even when we combine them in stages.

Adding linear combinations still produces a linear combination.

The Column Space

Vector Space

For a nonempty subset of $\mathbb{R}^n$ with the usual operations, check two properties.

CLOSED UNDER ADDITION

If u and v are in the space, then $u + v$ is also in the space.

$$(1, 0) + (2, 0) = (3, 0)$$

CLOSED UNDER SCALAR MULTIPLICATION

If v is in the space, then cv is in the space for every real scalar c .

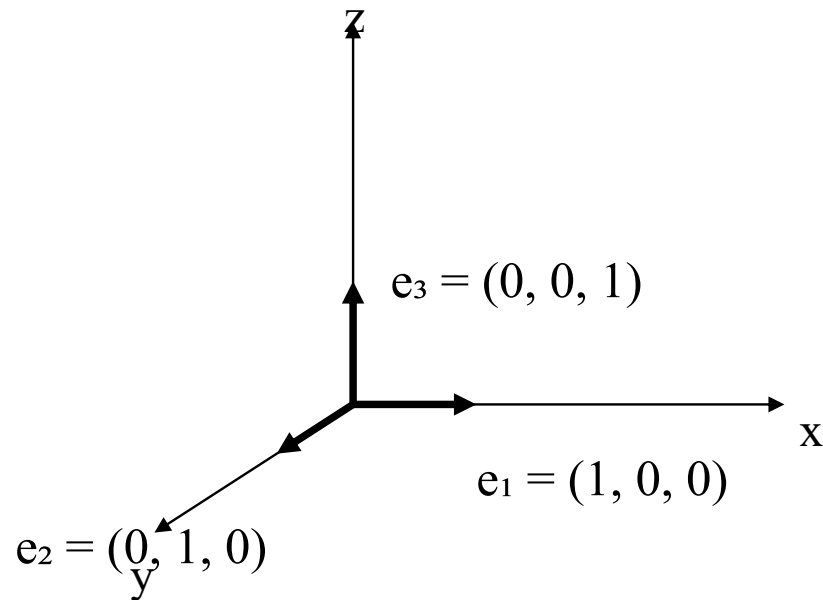
$$-2(1, 0) = (-2, 0)$$

The x-axis satisfies both rules and contains the zero vector.

With the usual operations, combinations of vectors must stay in the same space.

The Column Space

Example: $\mathbb{R}^3$



$$(a, b, c) = a e_1 + b e_2 + c e_3$$

Each coefficient chooses how far to move along one coordinate axis.

$$(2, 5, 1) = 2e_1 + 5e_2 + e_3$$

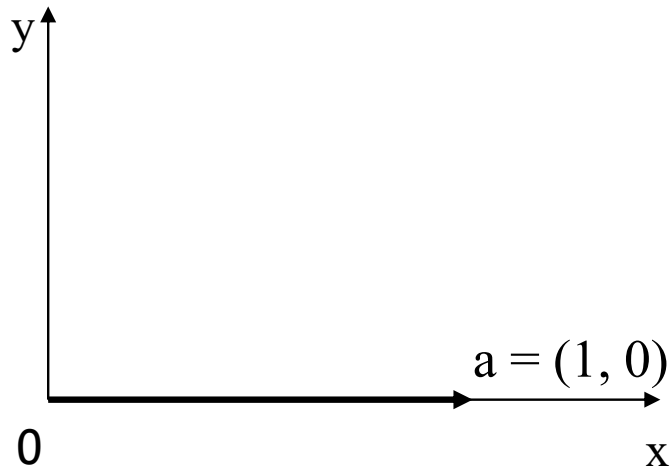
Any real values of a , b , and c are allowed.

Three independent coordinate directions can generate every vector in $\mathbb{R}^3$.

The Column Space

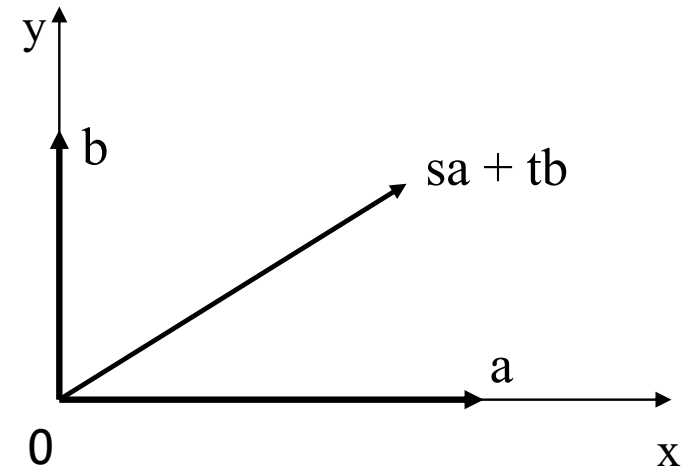
Span

ONE DIRECTION



Multiples of a reach only the x-axis.
Adding more multiples does not create
a new direction.

TWO INDEPENDENT DIRECTIONS



Combinations of a and b reach the
whole plane because the directions are
independent.

Span means all vectors that can be generated by linear combinations.

The Column Space

Linear Independence

$$a = (1, 0) \quad b = (0, 1) \quad c = (2, 3)$$

$$c = 2a + 3b$$

The vector c can already be generated from a and b , so it adds no new direction.

The set $\{a, b, c\}$ is dependent. Removing c does not change its span.

The pair $\{a, b\}$ is independent: neither vector is a multiple of the other.

A set is linearly independent when no vector is redundant.

The Column Space

Basis

Basis = Linearly Independent + Spans the Space

Candidate set	Independent?	Spans $\mathbb{R}^2$?
$a = (1, 0)$ and $b = (0, 1)$	Yes	Yes
$a, b,$ and $c = (2, 3)$	No	Yes
a only	Yes	No

The pair $\{a, b\}$ is a basis of $\mathbb{R}^2$. The extra vector c is unnecessary.

The number of vectors in a basis is the dimension of the space.

A basis spans the space without including redundant vectors.

The Column Space

Example: Redundant Features

Each lizard is one observation. Each column describes one feature.

Lizard	Length (m)	Weight (kg)	Speed (kph)	Age (years)
Gecko Grayson	0.5	0.1	4	10
Komodo Ken	2.5	50	15	20
Iguana Ian	0.4	9	30	15

For example, Komodo Ken has a recorded weight of 50 kg.

Question: Would another column for weight in pounds add new information?

A new column does not always provide new information.

The Column Space

Same Weight, Different Units

Lizard	Weight (kg)	Weight (lb)
Gecko Grayson	0.1	0.22
Komodo Ken	50	110
Iguana Ian	9	19.8

Weight in lb $\approx 2.2 \times$ Weight in kg

For example, 50 kg becomes approximately 110 lb using the rounded factor.

One weight column determines the other, so the second column is redundant.

Changing the unit does not create a new independent feature.

The Column Space

Why Does This Matter in Data Science?

16 STORED VALUES

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

4 VALUES + A REPEAT RULE

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Repeat the same row four times.

This simple array can be reconstructed exactly from four values and the rule.

Real images may contain many pixel values without this exact repetition.

Fewer coordinates may be enough when some information is redundant.

The Column Space

Worked Example: Reachable Outputs

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$A v = (v_1 + 2v_2, 0)$$

REACHABLE

(3, 0) is reachable. Choose $v = (1, 1)$, since $1 + 2 \times 1 = 3$.

NOT REACHABLE

(0, 1) is not reachable. The second output is always zero.

Therefore, $C(A)$ is the x-axis, with one possible basis vector (1, 0).

Dependent columns restrict the set of outputs that can be produced.

The Fundamental Spaces

The Null Space

$$A v = 0$$

1. Identify the inputs

Find every vector that the matrix maps to the zero vector.

2. Interpret the result

A nonzero input can disappear when its contributions cancel or are removed.

Instead of asking what comes out, ask which inputs vanish.

The Null Space

Null Space

$$N(A) = \{v : A v = 0\}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A NONZERO INPUT DISAPPEARS

The matrix keeps the first coordinate and removes the second. Thus (0, 5) maps to zero.

Every input of the form (0, t) also maps to zero.

The zero vector is always included; this example also has nonzero null-space vectors.

The null space contains all input vectors that A maps to zero.

The Null Space

Orthogonality and Null Space

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

CHECK EACH ROW

$$(1, 0, 0) \cdot (0, 0, 4) = 0$$

$$(0, 1, 0) \cdot (0, 0, 4) = 0$$

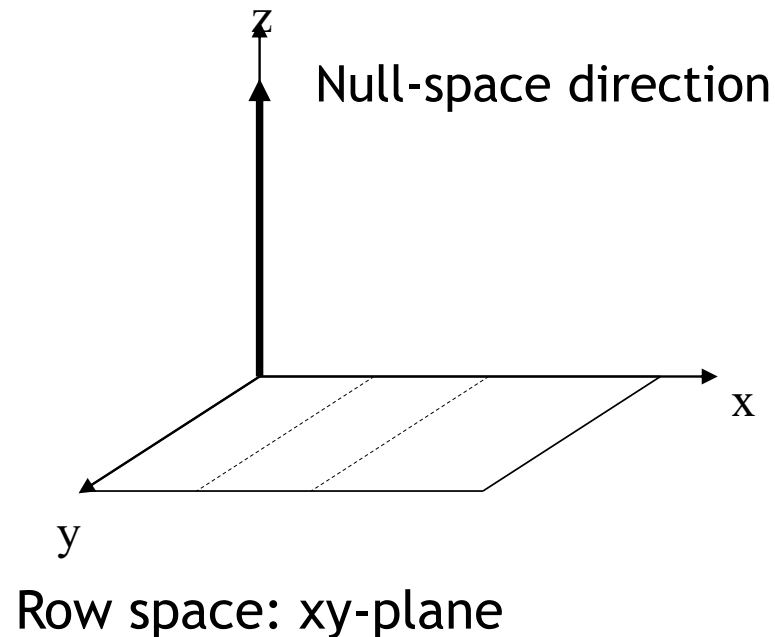
Each output entry is the dot product of one row with v .

Both entries are zero. Therefore, v is orthogonal to both rows and belongs to $N(A)$.

$Av = 0$ means that v has a zero dot product with every row of A .

The Null Space

Geometric Interpretation



TWO INDEPENDENT ROWS

The rows $(1, 0, 0)$ and $(0, 1, 0)$ span the xy-plane, not all of $\mathbb{R}^3$.

PERPENDICULAR INPUTS

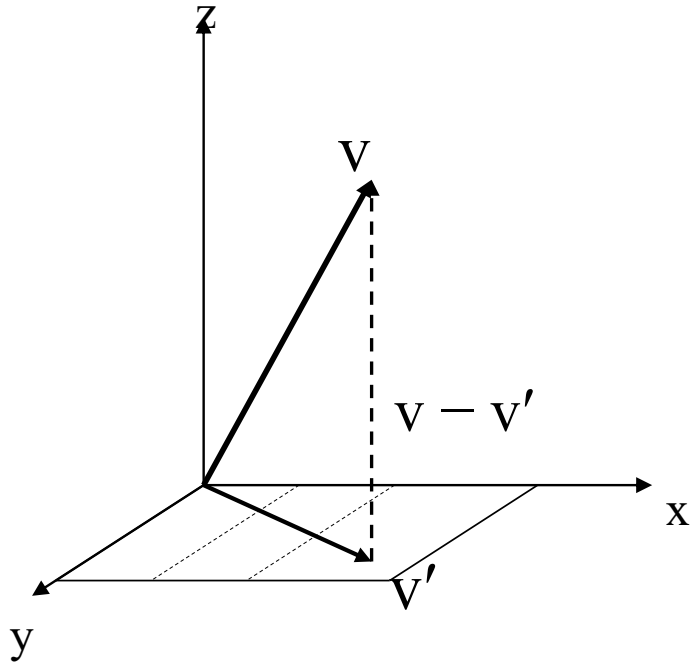
Vectors along the z-axis have a zero dot product with every vector in that plane.

In other 2×3 matrices, dependent or zero rows may span only a line or the origin.

Two row vectors cannot span the whole three-dimensional input space.

The Null Space

Finding a Vector in the Null Space



1. Choose a vector outside the plane.

$$V = (3, 2, 4)$$

2. Project it onto the xy-plane.

$$V' = (3, 2, 0)$$

3. Subtract the projection.

$$V - V' = (0, 0, 4)$$

The remaining vector is perpendicular to the row space, so it belongs to $N(A)$.

Subtract the row-space projection to obtain a null-space vector.

The Null Space

Nontrivial Null Space

TRIVIAL NULL SPACE

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$N(A) = \{0\}$$

Only the zero input maps to zero. The identity matrix preserves both coordinates.

NONTRIVIAL NULL SPACE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A(0, t) = (0, 0)$$

The second coordinate disappears. Every $(0, t)$ maps to zero, including nonzero inputs.

A nontrivial null space contains at least one nonzero vector.

The Null Space

Null Space and Matrix Inverse

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A(2, 1) = (2, 0)$$

$$A(2, 9) = (2, 0)$$

Two different inputs produce exactly the same output.

From $(2, 0)$ alone, we cannot tell whether the second coordinate was 1, 9, or another value.

The transformation cannot be uniquely undone, so this square matrix is singular.

For a square matrix, a nontrivial null space means that no inverse exists.

The Fundamental Spaces

Column Space and Null Space

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

A maps a 2D input vector to a 2D output vector.

COLUMN SPACE

$$C(A) = \text{span}\{(1, 0)\}$$

Every output has the form $(s, 0)$, so the output space is the x-axis.

NULL SPACE

$$N(A) = \text{span}\{(-2, 1)\}$$

For $v = t(-2, 1)$, the terms $-2t$ and $2t$ cancel, so $Av = 0$.

The same matrix has an output space and a space of disappearing inputs.